

REFLECTIONLESS SPONGE LAYERS AS ABSORBING BOUNDARY CONDITIONS FOR THE NUMERICAL SOLUTION OF MAXWELL EQUATIONS IN RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATES*

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Abstract. A scaling argument is used to derive reflectionless sponge layers to absorb outgoing time-harmonic waves in numerical solutions of the three-dimensional elliptic Maxwell equations in rectangular, cylindrical, and spherical coordinates. We also develop our reflectionless sponge layers to absorb outgoing transient waves in numerical solutions of the time-domain Maxwell equations and prove that these absorbing layers are described by causal, strongly well-posed hyperbolic systems. A representative result is given for wave scattering by a compact obstacle to demonstrate the many orders of magnitude improvement offered by our approach over standard techniques for computational domain truncation.

Key words. perfectly matched layers, absorbing boundary conditions, Maxwell equations

AMS subject classifications. 35L05, 65M99, 78A40, 78A45

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1. Introduction. We wish to solve the time-dependent Maxwell equations

$$(1.1) \quad \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \nabla \cdot \mathbf{D} = 0, \\ \frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \mathbf{H}, \nabla \cdot \mathbf{B} = 0, \end{aligned}$$

closed with constitutive relations $\mathbf{D} = \epsilon(\mathbf{x})\mathbf{E}$, $\mathbf{B} = \mu(\mathbf{x})\mathbf{H}$, over a computational domain $\Omega_c \subset \mathcal{R}^3$, with boundary $\partial\Omega_c$, embedded in an infinite homogeneous dielectric background medium Ω_m of constant permittivity $\epsilon(\mathbf{x}) = \epsilon$ and permeability $\mu(\mathbf{x}) = \mu$. At $t = 0$ the fields are zero everywhere. An external forcing of (1.1) represents incident transient electromagnetic waves that excite fields scattered by obstacles in Ω_c and which satisfy a Sommerfeld radiation condition at infinity. This well-posed (symmetric) hyperbolic initial boundary value problem is to be solved with a numerical scheme.

In order to complete the interior numerical scheme on $\partial\Omega_c$, an absorbing boundary condition (ABC) must be imposed there. At the same time, the ABC must accurately represent waves that propagate to infinity once they enter Ω_m . An approximation of the pseudo-differential operator that describes outgoing waves on $\partial\Omega_c$ is typically employed; a multitude of such conditions have been derived and implemented by many researchers, and are extensively reviewed in [9, 12]. In addition to approximate methods, exact ABC formulations are available and show much promise [13, 14, 15, 16].

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An alternative to ABCs is to surround Ω_c with a semi-infinite wave absorbing layer $\Omega_l \subset \Omega_m$. Ideally, the transition from Ω_c to Ω_l should not reflect waves impinging on $\partial\Omega_c$, while the fields that enter Ω_l should attenuate as they propagate: first outward, toward $\partial\Omega_l$, and then inward, after reflection from the boundary condition imposed on $\partial\Omega_l$. We call this set of attributes “ideal properties.” For use as an ABC, the layer that possesses these ideal properties is typically terminated by applying a boundary condition on the tangential electric fields at the outer boundary $\partial\Omega_l$, i.e., $\mathbf{B}(\hat{\mathbf{n}} \times \mathbf{E}) = 0$. The approach herein is independent of the method used to discretize the problem in Ω_c and our results can be employed to truncate infinite domains in which (1.1) is solved with finite element, finite difference, or spectral methods.

Previous work with sponge layers [10, 11] did not achieve these ideal properties. Karni [25] derived and demonstrated a reflectionless sponge layer that exhibits these properties for a scalar plane wave propagating in a given direction, but we would like to absorb vector waves propagating in all possible directions simultaneously. The existence of sponge layers that exhibit these ideal properties was shown by Berenger [1], who constructed the first “split-field” perfectly matched layer (PML) and used it as an ABC. Unfortunately, this very successful sponge layer introduces nonphysical dependent variables in the time domain through an arbitrary splitting of the fields that increases the dimension of the hyperbolic Maxwell system. The Berenger PML does not correspond to a physical process, and the mathematical manifestation of this is the weak well-posedness of the relevant Cauchy problem. It was shown in [3] that the hyperbolic Berenger system is not symmetric, that it allows wave modes which grow linearly, in time and that it is ill-posed under perturbations.

Well-posed alternatives do exist. In the engineering literature they are referred to as “unsplit-field PML formulations,” and some approaches in rectangular coordinates [2, 4, 17, 18] have become popular for the numerical solution of (1.1). Fourth-order accurate discretizations of such reflectionless sponge layers for use with high-order finite difference schemes have also been analyzed and implemented to solve scattering problems [19]. Additionally, reflectionless sponge layers have been demonstrated as very effective when (1.1) is solved with spectral methods [20, 21].

Herein, we construct well-posed sponge layers for the three-dimensional Maxwell equations in cylindrical and spherical coordinates and, in keeping with established terminology, we call them “reflectionless sponge layers.” Also, we prove they possess the ideal properties required for use as an ABC. Our derivation begins in the frequency domain, i.e., with the elliptic form of (1.1) obtained after applying the Fourier transform in the time direction. Our method may be viewed as a similarity transformation that maps a homogeneous isotropic dielectric to an inhomogeneous uniaxially anisotropic dielectric that exhibits the ideal properties. Section 2 describes our procedure in the three-dimensional cylindrical and spherical coordinates. For the purpose of demonstrating the effectiveness of our layer we employ it as an ABC and obtain the exact solution of the problem of time-harmonic plane wave scattering from a metal cylinder embedded in a finite-sized domain in two-dimensional cylindrical coordinates. This solution is then compared with that obtained when the cylinder is embedded in an unbounded domain. We also compute the finite-domain exact solution obtained by using a popular local ABC [27] on $\partial\Omega_c$ instead of the sponge layer and compare it with the other two. A greater-than-four-orders-of-magnitude improvement over standard domain truncation techniques is obtained. In section 3, the hyperbolic form of the equations describing the action of the layer on transient waves is derived and proofs of hyperbolicity, causality, and well-posedness are given. Section 4 closes the

paper with a summary, some comments regarding generalizations of the approach to more complicated propagation problems, and a description of future work. The case of rectangular coordinates is given in the appendix.

2. The method. We begin with the three-dimensional elliptic Maxwell equations (assuming $e^{-i\omega t}$ dependence for the fields in (1.1)),

$$(2.1) \quad \begin{aligned} -i\omega\epsilon(\mathbf{x}')\mathbf{E}' &= \nabla' \times \mathbf{H}', & \nabla' \cdot (\epsilon(\mathbf{x}')\mathbf{E}') &= 0, \\ -i\omega\mu(\mathbf{x}')\mathbf{H}' &= -\nabla' \times \mathbf{E}', & \nabla' \cdot (\mu(\mathbf{x}')\mathbf{H}') &= 0, \end{aligned}$$

in an inhomogeneous isotropic (herein lossless) dielectric that fills all of $\mathcal{R}'^3$. This space is then partitioned as follows: the volume Ω_c , identified in applications with the interior computational domain where scatterers are embedded and $\lim_{\mathbf{x}' \rightarrow \partial\Omega_c^-} \epsilon(\mathbf{x}') = \epsilon$ and $\lim_{\mathbf{x}' \rightarrow \partial\Omega_c^-} \mu(\mathbf{x}') = \mu$, and the volume Ω_m (the tail), which extends to infinity and $\epsilon(\mathbf{x}') = \epsilon$ and $\mu(\mathbf{x}') = \mu$. Waves entering Ω_m must propagate undisturbed to infinity. Next, we scale the independent and dependent variables in (2.1) with the (similarity) transformations

$$(2.2) \quad \begin{aligned} \mathbf{x}' &= \begin{cases} \mathbf{x}; & \mathbf{x}' \in \Omega_c, \\ \mathbf{S}(\mathbf{x}, \omega) \cdot \mathbf{x}; & \mathbf{x}' \in \Omega_m \cup \partial\Omega_c, \end{cases} \\ \mathbf{E}' &= \begin{cases} \mathbf{E}; & \mathbf{x}' \in \Omega_c, \\ \mathbf{A}^e(\mathbf{x}, \omega) \cdot \mathbf{E}; & \mathbf{x}' \in \Omega_m \cup \partial\Omega_c, \end{cases} \\ \mathbf{H}' &= \begin{cases} \mathbf{H}; & \mathbf{x}' \in \Omega_c, \\ \mathbf{A}^m(\mathbf{x}, \omega) \cdot \mathbf{H}; & \mathbf{x}' \in \Omega_m \cup \partial\Omega_c, \end{cases} \end{aligned}$$

where $\mathbf{x} \in \mathcal{R}^3$ is an independent variable with units of space, and $\omega \in \mathcal{R}$ is the frequency. In this paper we will show that a reflectionless region can be realized in cylindrical, spherical, and rectangular coordinates if the diagonal matrices $\mathbf{S}$, $\mathbf{A}^e$, and $\mathbf{A}^m$ are chosen so that $\mathbf{S} = \mathbf{I}$ for $\mathbf{x} \in \partial\Omega_c$ and coordinate-independent expressions in $\Omega_m \cup \partial\Omega_c$, such as $\nabla' \times \mathbf{E}'$ and $\nabla' \times \mathbf{H}'$, are invariant up to an overall factor that depends on $(\mathbf{x}, \omega)$; such a factor can be identified with electric and magnetic “material” properties. Throughout the paper, subscripts indicate the corresponding spatial direction, and bold-faced quantities represent three-dimensional vectors.

The elements of (2.2) involve the coordinate stretching $u' = \gamma_u(u, \omega)u$, where

$$(2.3) \quad \gamma_u(u, \omega) = \frac{u_0 + \int_{u_0}^u \alpha_u(s, \omega) ds}{u},$$

and $u \geq u_0 \in \mathcal{R}^+$ is the corresponding coordinate variable. Thus, derivatives in $\Omega_m \cup \partial\Omega_c$ transform according to $\frac{\partial}{\partial u'} = \zeta_u \frac{\partial}{\partial u}$, where $\zeta_u = \frac{1}{\alpha_u(u, \omega)}$. Note $\lim_{u \rightarrow u_0^+} \gamma_u = 1$ and $\lim_{u \rightarrow u_0^+} \frac{\partial u'}{\partial u} = \alpha_u(u_0, \omega)$. Our method is independent of the choice for $\alpha_u(u, \omega)$; there are many possibilities depending on the frequency content of the application at hand. Herein, in order to obtain frequency-independent exponential damping of propagating waves in all three coordinate systems we will choose $\alpha_u(u, \omega) \in \mathcal{C}$ with $\text{Im}\{\alpha_u(u, \omega)\} \approx O(\frac{1}{\omega}) > 0$, i.e.,

$$(2.4) \quad \alpha_u(s, \omega) = \xi_u \left(1 + \frac{i\sigma_u(s)}{\omega} \right), \quad xi_u \geq 1,$$

where $\sigma_u(s) = \sigma_u^{max}(\frac{s-s_0}{d_u})^n$; $n \geq 0$, $\xi_u \in \mathcal{R}$, $n \in \mathcal{I}$, $\sigma_u^{max} \in \mathcal{R}^+$, and d_u is a length scale in the u -direction. Later, d_u will be the width of the sponge layer, and the functions $\sigma_u^e(u) = \epsilon\sigma_u(u)$ and $\sigma_u^m(u) = \mu\sigma_u(u)$ have units of electric and magnetic conductivity, respectively. The real constant $\xi_u (\geq 1)$ allows for bending the waves toward the normal direction into the layer by increasing the corresponding component of the wave vector while leaving the tangential component unchanged; at the same time, it increases the damping in the normal direction into the layer. If $\sigma_u(u_0^+) = 0$, then $\lim_{u \rightarrow u_0^+} \frac{\partial u'}{\partial u} = \xi_u$. Experience has shown the choice (2.4) to be best suited for problems in which the fields have no zero-frequency content. For problems with zero-frequency content one may choose $\alpha_u(s, \omega) = \xi_u(1 + \sigma_u(s)/(1 - i\omega))$ which is regular (and real) at $\omega = 0$; we discuss this further in (A.6).

When (2.4) is used, the independent variables in (2.1) may also be viewed as being analytically continued into the space of complex numbers in $\Omega_m \cup \partial\Omega_c$, while $\mathbf{x}' \in \mathcal{R}^3$ in Ω_c ; this incomplete point of view, central to [5, 6], necessitates an arbitrary splitting of the fields to obtain the hyperbolic (time-domain) formulation which then shares the undesirable mathematical properties of Berenger's PML [3]. Our approach does not suffer in this manner.

For numerical implementation as an ABC, the unbounded absorbing region Ω_m must be truncated with a boundary condition at some finite distance from $\partial\Omega_c$ to produce the layer Ω_l . A homogeneous Dirichlet condition on the electric fields tangential to $\partial\Omega_l$ will be used herein, i.e., $\mathbf{B} = \mathbf{I}$ (but other options are available [23]), and we will show that the resultant nonzero reflection coefficient is exponentially small.

2.1. Cylindrical (ρ, ϕ, z) coordinates. In this coordinate system, Maxwell's equations (2.1) in $\Omega_m = \mathcal{R}^3 \setminus \Omega_c$ are

$$(2.5) \quad \begin{aligned} -i\omega\epsilon(E_{\rho'}, E_{\phi'}, E_{z'}) &= \frac{1}{\rho'} \begin{vmatrix} \hat{\mathbf{a}}_{\rho'} & \rho' \hat{\mathbf{a}}_{\phi'} & \hat{\mathbf{a}}_{z'} \\ \frac{\partial}{\partial \rho'} & \frac{\partial}{\partial \phi'} & \frac{\partial}{\partial z'} \\ H_{\rho'} & (\rho' H_{\phi'}) & H_{z'} \end{vmatrix}, \\ -i\omega\mu(H_{\rho'}, H_{\phi'}, H_{z'}) &= -\frac{1}{\rho'} \begin{vmatrix} \hat{\mathbf{a}}_{\rho'} & \rho' \hat{\mathbf{a}}_{\phi'} & \hat{\mathbf{a}}_{z'} \\ \frac{\partial}{\partial \rho'} & \frac{\partial}{\partial \phi'} & \frac{\partial}{\partial z'} \\ E_{\rho'} & (\rho' E_{\phi'}) & E_{z'} \end{vmatrix}, \end{aligned}$$

and the fields are divergence-free, i.e., $\nabla' \cdot (\epsilon \mathbf{E}') = 0$, $\nabla' \cdot (\mu \mathbf{H}') = 0$. The volume Ω_c occupies the region $0 \leq \rho' < \rho_0$, $0 \leq \phi' < 2\pi$, $|z'| < z_0$. We identify three distinct subregions of Ω_m for which sponge equations must be derived (see Figure 2.1).

Case a. Ω_ρ subregion: $\rho' \geq \rho_0$, $0 \leq \phi' < 2\pi$, $|z'| \leq z_0$.

The presence of $\rho' H_{\phi'}$ and $\rho' E_{\phi'}$ in the right-hand side (r.h.s.) of (2.5) requires individual entries in these products to scale accordingly in order for the physical units of the fields (volts and amperes) to be preserved. We are naturally led to the following form for the scaling (2.2):

$$(2.6) \quad \begin{aligned} (\rho', \phi', z')^T &= \text{diag}\{\gamma_\rho, 1, 1\} \cdot (\rho, \phi, z)^T, \\ \mathbf{E}' &= \text{diag}\left\{\zeta_\rho, \frac{1}{\gamma_\rho}, 1\right\} \cdot \mathbf{E}, \\ \mathbf{H}' &= \text{diag}\left\{\zeta_\rho, \frac{1}{\gamma_\rho}, 1\right\} \cdot \mathbf{H}. \end{aligned}$$

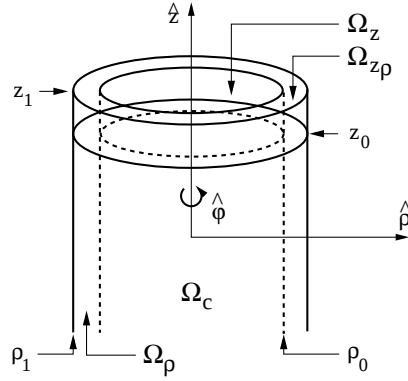


FIG. 2.1. Cylindrical coordinates.

Applying (2.6) to (2.5) we obtain

$$(2.7) \quad \begin{aligned} -i\omega\epsilon \begin{pmatrix} \gamma_\rho \zeta_\rho & 0 & 0 \\ 0 & \frac{1}{\gamma_\rho \zeta_\rho} & 0 \\ 0 & 0 & \frac{\gamma_\rho}{\zeta_\rho} \end{pmatrix} \cdot \mathbf{E} &= \nabla \times \mathbf{H}, \\ -i\omega\mu \begin{pmatrix} \gamma_\rho \zeta_\rho & 0 & 0 \\ 0 & \frac{1}{\gamma_\rho \zeta_\rho} & 0 \\ 0 & 0 & \frac{\gamma_\rho}{\zeta_\rho} \end{pmatrix} \cdot \mathbf{H} &= -\nabla \times \mathbf{E}. \end{aligned}$$

The diagonal “material” tensor in the l.h.s. of (2.7) is denoted $\mathcal{T}$, and applying (2.6) to the divergence-free conditions we obtain, e.g., for the electric field,

$$(2.8) \quad \begin{aligned} \nabla' \cdot (\epsilon \mathbf{E}') &= 0 \\ \rightarrow \epsilon \left(\frac{1}{\gamma_\rho \rho} \zeta_\rho \frac{\partial}{\partial \rho} (\rho \gamma_\rho \zeta_\rho E_\rho) + \frac{1}{\rho \gamma_\rho^2} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right) &= \nabla \cdot (\epsilon \mathcal{T} \cdot \mathbf{E}) = 0. \end{aligned}$$

Similarly, for the magnetic field, $\nabla' \cdot (\mu \mathbf{H}') = 0 \rightarrow \nabla \cdot (\mu \mathcal{T} \cdot \mathbf{H}) = 0$. The form of the mapped divergence-free conditions allows $\epsilon \mathcal{T}$ and $\mu \mathcal{T}$, respectively, to be identified as uniaxial electric permittivity and magnetic permeability tensors. This facilitates the derivation in section 3 of the hyperbolic equations needed to implement the layer in the time domain.

The spatial scaling in (2.6) is continuous at $\rho = \rho_0$, while the field scaling (second and third equations) enforces the continuity of the unprimed tangential and primed normal electromagnetic field components (recall, $\lim_{u \rightarrow u_0^+} \gamma_u = 1$ and $u = \rho$); the transition from Ω_c to Ω_ρ should be reflectionless.

We now prove that $\partial \Omega_c$ is reflectionless for propagating waves impinging on it from the interior, and that, once (2.4) is chosen, these perfectly transmitted waves decay exponentially, independently of the frequency ω , as they propagate to infinity in Ω_ρ .

First, we observe that for constant ϵ, μ the vector Helmholtz equation for the electric field, $\omega^2 \epsilon \mu \mathbf{E}' = \nabla' \times \nabla' \times \mathbf{E}'$, is obtained after the magnetic field is eliminated from (2.5). In cylindrical coordinates, and using $\nabla' \cdot \mathbf{E}' = 0$, this vector equation decomposes into two coupled equations for $E_{\rho'}$ and $E_{\phi'}$, while the $E_{z'}$ component

uncouples and satisfies the scalar Helmholtz equation

$$(2.9) \quad \frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial E_{z'}}{\partial \rho'} \right) + \frac{1}{\rho'^2} \frac{\partial^2 E_{z'}}{\partial \phi'^2} + \frac{\partial^2 E_{z'}}{\partial z'^2} + \omega^2 \epsilon \mu E_{z'} = 0.$$

This is similar for $\mathbf{H}'$ using $\nabla' \cdot \mathbf{H}' = 0$. The solutions of (2.9), along with those of the corresponding equation for $H_{z'}$, define TM-to- $\hat{z}$ ($H_{z'} = 0$) and TE-to- $\hat{z}$ ($E_{z'} = 0$) orthogonal polarizations. An arbitrary propagating electromagnetic wave with wave vector $\mathbf{k}$ is represented as a linear combination of these two polarizations. To determine the reflection properties of $\partial\Omega_c$ we will need both TE and TM waves to match the electromagnetic boundary conditions across the cylindrical surface at $\rho = \rho_0$. All components of TE/TM outgoing waves in Ω_m are solutions of (2.9) and they are expressed as a Fourier series in the angular coordinate

$$(2.10) \quad U_{z'}(\rho', \phi', z') = \sum_{m=-\infty}^{\infty} U_z^m(\rho', \phi', z') = \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(k_\rho \rho') e^{im\phi' + i\tilde{\kappa}_z z'},$$

where $k_\rho = \sqrt{\omega^2 \epsilon \mu - \tilde{\kappa}_z^2}$, $\tilde{\kappa}_z = \hat{z} \cdot \mathbf{k}$, $m \in (-\infty, \infty)$ is an integer, and $H_m^{(1)}(\cdot)$ is the Hankel function of the first kind and of integer order m .

Dropping the primes in Ω_c and suppressing the $e^{i\tilde{\kappa}_z z + im\phi}$ dependence, the $\hat{z}$ -directed fields are

$$(2.11) \quad \begin{pmatrix} E_z^1 \\ H_z^1 \end{pmatrix} = \{H_m^{(1)}(k_\rho \rho) \mathbf{I} + J_m(k_\rho \rho) \mathbf{R}_{12}\} \cdot \begin{pmatrix} E_0 \\ H_0 \end{pmatrix},$$

where the first term represents a wave that is outgoing in Ω_c , the second term represents the standing wave generated by the reflection of the outgoing wave at $\rho = \rho_0$ and the requirement for finite reflected fields at $\rho = 0$, $J_m(\cdot)$ is the Bessel function of integer order, $(E_0, H_0)^T$ is an arbitrary complex vector of incident field amplitudes, and $\mathbf{R}_{12}$ is the 2×2 reflection matrix (generated by the presence of $\partial\Omega_c$), which we must prove is the null matrix. The off-diagonal terms of $\mathbf{R}_{12}$ represent possible TE–TM mode conversion at $\rho = \rho_0$. Continuity of tangential fields at $\rho = \rho_0$ will also require the ϕ -directed fields in Ω_c :

$$(2.12) \quad \begin{pmatrix} H_\phi^1 \\ E_\phi^1 \end{pmatrix} = \frac{1}{k_\rho^2} \{ \mathbf{H}_m^{(1)}(k_\rho \rho) + \mathbf{J}_m(k_\rho \rho) \cdot \mathbf{R}_{12} \} \cdot \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}.$$

The matrices $\mathbf{H}_m^{(1)}(k_\rho \rho)$ and $\mathbf{J}_m(k_\rho \rho)$ in (2.12) are obtained by substituting the indicated Hankel or Bessel function in

$$(2.13) \quad \mathbf{U}_m(k_\rho \rho) = \begin{pmatrix} i\omega \epsilon k_\rho \rho U'(k_\rho \rho) & -m\tilde{\kappa}_z U(k_\rho \rho) \\ -m\tilde{\kappa}_z U(k_\rho \rho) & -i\omega \mu k_\rho \rho U'(k_\rho \rho) \end{pmatrix},$$

where the prime denotes differentiation with respect to the argument of the function [26].

To determine the fields in the sponge region and allow the use of the standard results for cylindrical waves given above, we must show the decoupling of the $\hat{z}$ -component in the vector Helmholtz equation also occurs in Ω_ρ . To that end we use (2.8) to derive the Helmholtz equation for the E_z field component. Eliminating the magnetic field from (2.7), the vector Helmholtz equation for the electric field in Ω_ρ is

$$(2.14) \quad \omega^2 \epsilon \mu \mathbf{E} = \mathcal{T}^{-1} \cdot \nabla \times \mathcal{T}^{-1} \cdot \nabla \times \mathbf{E},$$

where $\mathcal{T}^{-1}$ is the inverse of the “material” tensor in (2.7). The z -component of (2.14) is

$$(2.15) \quad \omega^2 \epsilon \mu E_z = \frac{\zeta_\rho}{\rho \gamma_\rho} \left[\frac{\partial}{\partial \rho} \left\{ \zeta_\rho \rho \gamma_\rho \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \right) \right\} - \zeta_\rho \frac{\partial}{\partial \phi} \left\{ \frac{1}{\rho \gamma_\rho} \left(\frac{\partial E_z}{\partial \phi} - \frac{\partial(\rho E_\phi)}{\partial z} \right) \right\} \right].$$

It is clear from (2.8) that

$$(2.16) \quad \frac{\partial}{\partial z} \left[\frac{1}{\rho \gamma_\rho} \zeta_\rho \frac{\partial}{\partial \rho} (\zeta_\rho \rho \gamma_\rho E_\rho) + \frac{1}{\rho^2 \gamma_\rho^2} \frac{\partial E_\phi}{\partial \phi} \right] = -\frac{\partial^2 E_z}{\partial z^2},$$

so (2.15) becomes

$$(2.17) \quad \frac{1}{\rho \gamma_\rho} \zeta_\rho \frac{\partial}{\partial \rho} \left(\rho \gamma_\rho \zeta_\rho \frac{\partial E_z}{\partial \rho} \right) + \frac{1}{\rho^2 \gamma_\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \omega^2 \epsilon \mu E_z = 0.$$

We observe that, in light of (2.6), (2.17) is really (2.9). This also holds for the vector Helmholtz equation satisfied by the magnetic field.

Due to the translational invariance of the problem in the $\hat{z}$ -direction, the waves in Ω_ρ also have an $e^{i\tilde{\kappa}_z z + im\phi}$ dependence (which is suppressed below). Consequently, the solution of (2.17) (and of the corresponding one for H_z) is obtained after applying the spatial coordinate mapping to (2.10). Thus, the $\hat{z}$ -components of the outgoing fields in this region are

$$(2.18) \quad \begin{pmatrix} E_z^2 \\ H_z^2 \end{pmatrix} = H_m^{(1)}(k_\rho \gamma_\rho \rho) \mathbf{T}_{12} \cdot \begin{pmatrix} E_0 \\ H_0 \end{pmatrix},$$

where $\mathbf{T}_{12}$ is the 2×2 transmission matrix whose off-diagonal entries represent possible mode conversion on $\partial\Omega_c$. We must prove $\mathbf{T}_{12}$ is the identity matrix. Tangential field continuity at $\rho = \rho_0$ will require the $\hat{\phi}$ -directed fields in Ω_ρ . From (2.7), for waves with $e^{i\tilde{\kappa}_z z + im\phi}$ dependence, these are

$$(2.19) \quad \begin{pmatrix} H_\phi^2 \\ E_\phi^2 \end{pmatrix} = \frac{1}{k_\rho^2 \rho} \begin{pmatrix} i\omega \epsilon k_\rho \gamma_\rho \rho H^{(1)'}(k_\rho \gamma_\rho \rho) & -m\tilde{\kappa}_z H^{(1)}(k_\rho \gamma_\rho \rho) \\ -m\tilde{\kappa}_z H^{(1)}(k_\rho \gamma_\rho \rho) & -i\omega \mu k_\rho \gamma_\rho \rho H^{(1)'}(k_\rho \gamma_\rho \rho) \end{pmatrix} \cdot \mathbf{T}_{12} \cdot \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}.$$

Finally, it is trivial to determine that matching (2.11) to (2.18) and (2.12) to (2.19) at $\rho = \rho_0$ (using $\gamma_\rho(\rho_0) = 1$) leads to the desired result, i.e., $\mathbf{R}_{12} = \mathbf{0}$ and $\mathbf{T}_{12} = \mathbf{I}$. Indeed, $\partial\Omega_c$ is reflectionless for arbitrary waves propagating out of Ω_c .

We must also verify that (2.18), when (2.4) is used in (2.3) and (2.6), decays exponentially in the $\hat{\rho}$ -direction independently of the frequency, for an incident wave of given $\tilde{\kappa}_z = \omega \sqrt{\epsilon \mu} \cos \theta$, where θ is the angle between the $\hat{z}$ -axis and $\mathbf{k}$. Noting that $\rho' = \gamma_\rho(\rho, \omega) \rho$ exhibits a positive imaginary part, which varies as $O(\frac{1}{\omega})$, and using the asymptotic form of the outgoing Hankel function in (2.18), the desirable frequency-independent exponential decay is obtained in the $\hat{\rho}$ -direction

$$(2.20) \quad |H_m^{(1)}(\eta \rho')| \approx \frac{|C_m|}{\sqrt{\eta |\rho'|}} e^{-\eta \rho'}, \quad |\rho'| \rightarrow \infty,$$

where $\eta = \omega \sqrt{\epsilon \mu} \sin \theta$, and $\rho'_i = \frac{\xi_\rho}{\omega} \int_{\rho_0}^{\rho} \sigma_\rho(s) ds$ is the imaginary part of ρ' . The asymptotic condition $|\rho'| \rightarrow \infty$, while clearly achievable if $\rho \rightarrow \infty$ in the semi-infinite

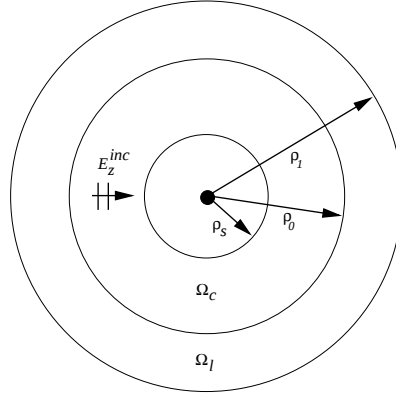


FIG. 2.2. Geometry of scattering problem in polar coordinates.

region Ω_m , can also be achieved if the product $\xi_\rho \sigma_\rho(\rho)$ is sufficiently large in the layer Ω_l of finite width $d_\rho = \rho_1 - \rho_0$. The decay (2.20) also applies to waves moving toward $\rho = \rho_0$ in the layer since they are proportional to $H_m^{(2)}(\eta\rho')$ and ρ decreases.

To terminate Ω_m and provide the layer Ω_l for numerical implementation a homogeneous Dirichlet condition is typically applied to the tangential fields E_z and E_ϕ at $\rho = \rho_1$ ($0 \leq \phi < 2\pi$, $|z| \leq z_0$). We now show that this induces a nonzero reflection coefficient whose exponentially small maximum amplitude is controlled by the “material” properties and width of the layer. For simplicity, consider the TM-polarization case in polar coordinates for the electromagnetic field components (E_z, H_ρ, H_ϕ) . Assuming Ω_l is illuminated by cylindrical waves, and using standard analysis, we obtain the reflection coefficient

$$(2.21) \quad R_m^{\Omega_\rho}(k_\rho \gamma_\rho(\rho_1)\rho_1) = -\frac{H_m^{(1)}(k_\rho \gamma_\rho(\rho_1)\rho_1)}{H_m^{(2)}(k_\rho \gamma_\rho(\rho_1)\rho_1)}.$$

The leading-order asymptotic form of the Hankel functions and (2.3)–(2.4) allow us to determine that for $m < |k_\rho \gamma_\rho(\rho_1)\rho_1|$ the amplitude of (2.21) is exponentially small and independent of m , i.e.,

$$(2.22) \quad |R^{\Omega_\rho}| \approx e^{-2\eta \frac{\xi_\rho}{\omega} \int_{\rho_0}^{\rho_1} \sigma_\rho(s) ds}.$$

The modes $m \geq |k_\rho \gamma_\rho(\rho_1)\rho_1|$ correspond to glancing incidence and are totally reflected (see (A.12) for the analogue in rectangular coordinates). Nevertheless, in three dimensions, these glancing waves are incident onto the remaining regions $\Omega_{z\rho}$ and Ω_z used to truncate the computational domain and are maximally absorbed.

To demonstrate the performance of the layer we next consider TM-polarized time-harmonic scattering. A plane wave,

$$(2.23) \quad E_z^{inc}(\rho, \phi) = e^{ikx} = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho) e^{im\phi},$$

is incident from the left onto a metal cylinder of radius ρ_s , embedded in a dielectric with permittivity ϵ and permeability μ (see Figure 2.2). The wave number is $k =$

$\omega\sqrt{\epsilon\mu}$. The total electric field is zero on the cylinder and the scattered field satisfies the Sommerfeld radiation condition at infinity. It is given by

$$(2.24) \quad E_z^{scat}(\rho, \phi) = - \sum_{m=-\infty}^{\infty} i^m \frac{J_m(k\rho_s)}{H_m^{(1)}(k\rho_s)} H_m^{(1)}(k\rho) e^{im\phi}.$$

The scattered field can be found as a Fourier series even when, instead of the Sommerfeld condition at infinity, we use our sponge layer (2.7) of width $d_\rho = \rho_1 - \rho_0$, reduced to polar coordinates with $\theta = \pi/2$, to truncate the problem at a finite distance from the scatterer. It is

$$(2.25) \quad \begin{aligned} \tilde{E}_z^{scat}(\rho, \phi) = & - \sum_{m=-\infty}^{\infty} i^m \frac{J_m(k\rho_s)}{W_m(k\rho_s, k\gamma_\rho(\rho_1)\rho_1)} \\ & \times \{H_m^{(1)}(k\gamma_\rho(\rho)\rho) + R_m^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)H_m^{(2)}(k\gamma_\rho(\rho)\rho)\} e^{im\phi}, \end{aligned}$$

where $\rho_s \leq \rho \leq \rho_1$, $\gamma_\rho(\rho) = 1$ for $\rho \leq \rho_0$,

$$(2.26) \quad W_m(k\rho_s, k\gamma_\rho(\rho_1)\rho_1) = H_m^{(1)}(k\rho_s) + R_m^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)H_m^{(2)}(k\rho_s),$$

and $R_m^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)$ is given by (2.21). On the other hand, when the local ABC [27]

$$(2.27) \quad B_2[\tilde{E}_z^{scat}] = \left(-ik + \frac{\partial}{\partial \rho} + \frac{1}{2\rho} - \frac{i}{8k\rho^2} - \frac{i}{2k\rho^2} \frac{\partial^2}{\partial \phi^2} \right) \tilde{E}_z^{scat} = 0$$

is used at $\rho = \rho_0$ instead of the Sommerfeld radiation condition at infinity, we obtain the following scattered field:

$$(2.28) \quad \tilde{E}_z^{scat}(\rho, \phi) = - \sum_{m=-\infty}^{\infty} i^m \frac{J_m(k\rho_s)}{W_m(k\rho_s, k\rho_0)} \{H_m^{(1)}(k\rho) + R_m(k\rho_0)H_m^{(2)}(k\rho)\} e^{im\phi},$$

where

$$(2.29) \quad W_m(k\rho_s, k\rho_0) = H_m^{(1)}(k\rho_s) + R_m(k\rho_0)H_m^{(2)}(k\rho_s),$$

and

$$(2.30) \quad R_m(k\rho_0) = - \frac{B_2[H_m^{(1)}(k\rho)e^{im\phi}]}{B_2[H_m^{(2)}(k\rho)e^{im\phi}]} \Big|_{\rho=\rho_0}.$$

Noting that (2.25) and (2.28) reduce to (2.24), when $R_m^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)$ and $R_m(k\rho_0)$ are sufficiently close to zero, we proceed to gauge the relative success of the two boundary conditions by comparing these two reflection coefficients. In (2.21) we choose $\xi_\rho = 1$ and $\theta = \pi/2$. The spatial variation of the damping is the same as that in (2.4), and the argument of the Hankel functions in (2.21) is $k\rho_1 + i \frac{\sigma_\rho^{max} d_\rho}{c(n+1)} = k\rho_1 + iD$. The layer width is chosen so $k(\rho_1 - \rho_0) = 1$ always; i.e., the layer is electrically thin.

In Figure 2.3 we plot the reflection coefficients (2.30) and (2.21) as a function of the cylindrical mode index m for $k\rho_0 = 9$ and three values of D . It is seen that both boundary closures do not absorb glancing modes. However, for sufficiently large D ,

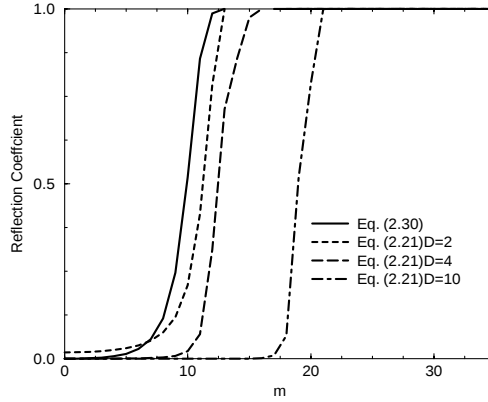


FIG. 2.3. The reflection coefficient as a function of the cylindrical mode index, m , for $k\rho_0 = 9$.

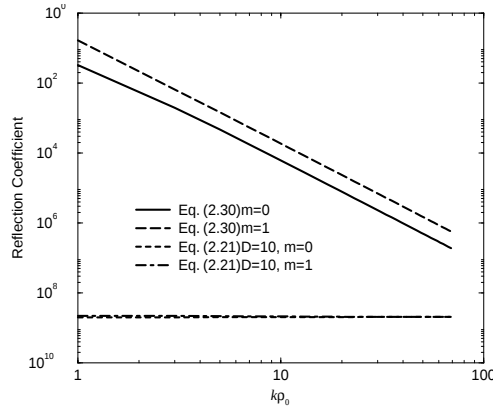


FIG. 2.4. The reflection coefficient as a function of $k\rho_0$ for the $m = 0, 1$ cylindrical modes.

the superiority of the sponge layer over the local ABC is evident for all propagating modes. The reflection coefficient for the $D = 10$ layer (for which $|k\gamma_\rho(\rho_1)\rho_1| \approx 15$) is effectively zero, i.e., exact, for the first 17 cylindrical modes. In contrast, the B_2 operator cannot be adjusted to absorb more propagating modes.

The behavior of the boundary closures as a function of the electrical distance from the origin to $\partial\Omega_c$ is compared in Figure 2.4. The first two cylindrical modes are considered but the same behavior is obtained for all higher modes below glancing. Again, our sponge layer outperforms the local ABC by many orders of magnitude. In the graph, the reflection coefficient $R_1^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)$ is only slightly higher than $R_0^{\Omega_\rho}(k\gamma_\rho(\rho_1)\rho_1)$. As expected, the sponge layer performance is insensitive to its location and deteriorates with increasing m at a rate that is much slower than that of the local ABC. In a full numerical simulation with the sponge layer as an ABC the amount of free space between the scatterer and $\partial\Omega_c$ that would require discretization is insignificant in comparison to the amount required when B_2 is used as an ABC instead.

Case b. $\Omega_{z\rho}$ subregion: $\rho' \geq \rho_0$, $0 \leq \phi' < 2\pi$, $|z'| \geq z_0$.

In this region we must introduce the scaling

$$(2.31) \quad \begin{aligned} (\rho', \phi', z')^T &= \text{diag}\{\gamma_\rho, 1, \gamma_z\} \cdot (\rho, \phi, z)^T, \\ \mathbf{E}' &= \text{diag}\left\{\zeta_\rho, \frac{1}{\gamma_\rho}, \zeta_z\right\} \cdot \mathbf{E}, \\ \mathbf{H}' &= \text{diag}\left\{\zeta_\rho, \frac{1}{\gamma_\rho}, \zeta_z\right\} \cdot \mathbf{H} \end{aligned}$$

into (2.5) and the divergence-free conditions. The transformed equations are similar to (2.7)–(2.8) but with “material” tensor

$$(2.32) \quad \mathcal{T} = \begin{pmatrix} \frac{\gamma_\rho \zeta_\rho}{\zeta_z} & 0 & 0 \\ 0 & \frac{1}{\gamma_\rho \zeta_\rho \zeta_z} & 0 \\ 0 & 0 & \frac{\gamma_\rho \zeta_z}{\zeta_\rho} \end{pmatrix}.$$

The scaling (2.31) enforces the continuity of the unprimed tangential and primed normal electromagnetic field components across the transition from Ω_z (see (2.35)) to $\Omega_{z\rho}$ at $\rho' = \rho_0$ and across the transition from Ω_ρ (see (2.6)) to $\Omega_{z\rho}$ at $|z'| = z_0$; these transitions are reflectionless.

A derivation similar to the one that led to (2.17), now leads to the following Helmholtz equation for the $\hat{z}$ -component of $\mathbf{E}$ in $\Omega_{z\rho}$:

$$(2.33) \quad \begin{aligned} \frac{1}{\rho\gamma_\rho} \zeta_\rho \frac{\partial}{\partial \rho} \left(\rho\gamma_\rho \zeta_\rho \frac{\partial(\zeta_z E_z)}{\partial \rho} \right) + \frac{1}{\rho^2 \gamma_\rho^2} \frac{\partial^2(\zeta_z E_z)}{\partial \phi^2} \\ + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial(\zeta_z E_z)}{\partial z} \right) + \omega^2 \epsilon \mu (\zeta_z E_z) = 0. \end{aligned}$$

Working backward with (2.31), we observe that (2.33) is really (2.9) written in the unprimed variables. These results also apply to the Helmholtz equation for the $\hat{z}$ -component of the magnetic field. Thus, the solution of (2.33), and of the corresponding equation for H_z , is obtained by substituting (2.31) into (2.10); using an argument similar to that in Ω_ρ we can prove both the $\Omega_z - \Omega_{z\rho}$ and $\Omega_\rho - \Omega_{z\rho}$ interfaces are reflectionless. Also, waves entering $\Omega_{z\rho}$ will decay exponentially independently of the frequency, now in both coordinate directions simultaneously. Since in this region both $\rho' = \gamma_\rho(\rho, \omega)\rho$ and $z' = \gamma_z(z, \omega)z$ exhibit positive imaginary parts that are $O(\frac{1}{\omega})$, the decay (2.20) is now

$$(2.34) \quad |H_m^{(1)}(\eta\rho')e^{i\tilde{\kappa}_z z'}| \approx \frac{|C_m|}{\sqrt{\eta|\rho'|}} e^{-\eta\rho'_i - \tilde{\kappa}_z z'_i}, \quad |\rho'| \rightarrow \infty, \quad |z'| \geq z_0,$$

where $z'_i = \frac{\xi_z}{\omega} \int_{z_0}^z \sigma_z(s) ds$ is the imaginary part of z' . Again, waves in the layer moving toward $|z| = z_0$ and $\rho = \rho_0$ are also exponentially damped. The Dirichlet condition is applied to the tangential fields E_z and E_ϕ at $\rho = \rho_1$ for $0 \leq \phi < 2\pi$, $z_0 \leq |z| \leq z_1$, and to E_ρ and E_ϕ at $|z| = z_1$ for $\rho_0 \leq \rho \leq \rho_1$, $0 \leq \phi < 2\pi$ to terminate the corner region for numerical implementation. The resulting reflection coefficients are exponentially small.

Case c. Ω_z subregion: $0 \leq \rho' \leq \rho_1$, $0 \leq \phi' < 2\pi$, $z_0 \leq |z'| \leq z_1$.

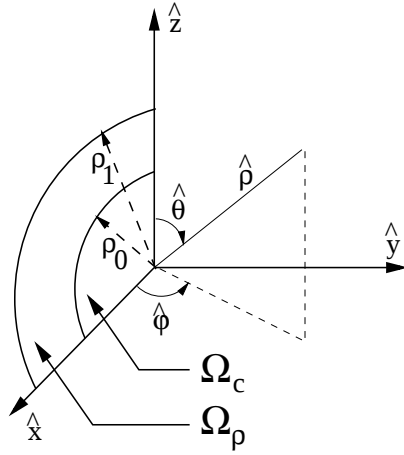


FIG. 2.5. Spherical coordinates.

In this region, the scaling

$$\begin{aligned}
 (\rho', \phi', z')^T &= \text{diag}\{1, 1, \gamma_z\} \cdot (\rho, \phi, z)^T, \\
 \mathbf{E}' &= \text{diag}\{1, 1, \zeta_z\} \cdot \mathbf{E}, \\
 \mathbf{H}' &= \text{diag}\{1, 1, \zeta_z\} \cdot \mathbf{H}
 \end{aligned}
 \tag{2.35}$$

is applied to (2.5) and the resulting $\mathcal{T}$ is given in (A.4). The sponge equations are those in (A.3) with the curl operator expressed in cylindrical coordinates. The region Ω_z is reflectionless for waves exiting Ω_c , and the penetrating waves decay as required (see (A.11)). The Dirichlet condition is applied to the tangential E_ρ and E_ϕ at $|z| = z_1$ for $0 \leq \rho \leq \rho_1$ to terminate the layer for numerical implementation; the resulting reflection coefficient (A.12) is exponentially small.

2.2. Spherical (ρ, θ, ϕ) coordinates. Here, the volume Ω_c is the interior of a sphere of radius ρ_0 , and Ω_m is the volume $\rho' \geq \rho_0$ which extends to infinity (see Figure 2.5). In light of Maxwell's equations in Ω_m ,

$$\begin{aligned}
 -i\omega\epsilon(E_{\rho'}, E_{\theta'}, E_{\phi'}) &= \frac{1}{\rho'^2 \sin \theta'} \begin{vmatrix} \hat{\mathbf{a}}_{\rho'} & \rho' \hat{\mathbf{a}}_{\theta'} & \rho' \sin \theta' \hat{\mathbf{a}}_{\phi'} \\ \frac{\partial}{\partial \rho'} & \frac{\partial}{\partial \theta'} & \frac{\partial}{\partial \phi'} \\ H_{\rho'} & (\rho' H_{\theta'}) & (\rho' \sin \theta' H_{\phi'}) \end{vmatrix}, \\
 -i\omega\mu(H_{\rho'}, H_{\theta'}, H_{\phi'}) &= -\frac{1}{\rho'^2 \sin \theta'} \begin{vmatrix} \hat{\mathbf{a}}_{\rho'} & \rho' \hat{\mathbf{a}}_{\theta'} & \rho' \sin \theta' \hat{\mathbf{a}}_{\phi'} \\ \frac{\partial}{\partial \rho'} & \frac{\partial}{\partial \theta'} & \frac{\partial}{\partial \phi'} \\ E_{\rho'} & (\rho' E_{\theta'}) & (\rho' \sin \theta' E_{\phi'}) \end{vmatrix},
 \end{aligned}
 \tag{2.36}$$

and the divergence-free conditions, $\nabla' \cdot (\epsilon \mathbf{E}') = 0$ and $\nabla' \cdot (\mu \mathbf{H}') = 0$, the appropriate scaling is

$$\begin{aligned}
 (\rho', \theta', \phi')^T &= \text{diag}\{\gamma_\rho, 1, 1\} \cdot (\rho, \theta, \phi)^T, \\
 \mathbf{E}' &= \text{diag}\{\zeta_\rho, \gamma_\rho^{-1}, \gamma_\rho^{-1}\} \cdot \mathbf{E}, \\
 \mathbf{H}' &= \text{diag}\{\zeta_\rho, \gamma_\rho^{-1}, \gamma_\rho^{-1}\} \cdot \mathbf{H}.
 \end{aligned}
 \tag{2.37}$$

Its application transforms (2.36) in Ω_ρ to

$$(2.38) \quad \begin{aligned} -i\omega\epsilon \begin{pmatrix} \gamma_\rho^2 \zeta_\rho & 0 & 0 \\ 0 & \frac{1}{\zeta_\rho} & 0 \\ 0 & 0 & \frac{1}{\zeta_\rho} \end{pmatrix} \cdot \mathbf{E} &= \nabla \times \mathbf{H}, \\ -i\omega\mu \begin{pmatrix} \gamma_\rho^2 \zeta_\rho & 0 & 0 \\ 0 & \frac{1}{\zeta_\rho} & 0 \\ 0 & 0 & \frac{1}{\zeta_\rho} \end{pmatrix} \cdot \mathbf{H} &= -\nabla \times \mathbf{E}. \end{aligned}$$

The diagonal matrix on the left-hand side (l.h.s.) of (2.38) is again labeled $\mathcal{T}$ and the divergence-free conditions for constant ϵ , μ transform as follows:

$$(2.39) \quad \begin{aligned} \nabla' \cdot (\epsilon \mathbf{E}') &= 0 \rightarrow \epsilon \left(\frac{1}{\gamma_\rho^2 \rho^2} \zeta_\rho \frac{\partial}{\partial \rho} (\rho^2 \gamma_\rho^2 \zeta_\rho E_\rho) + \frac{1}{\rho \gamma_\rho^2 \sin \theta} \frac{\partial (\sin \theta E_\theta)}{\partial \theta} + \frac{1}{\rho \gamma_\rho^2 \sin \theta} \frac{\partial E_\phi}{\partial \phi} \right) \\ &= \nabla \cdot (\epsilon \mathcal{T} \cdot \mathbf{E}) = 0. \end{aligned}$$

This is similar for the magnetic field relation $\nabla' \cdot (\mu \mathbf{H}') = 0 \rightarrow \nabla \cdot (\mu \mathcal{T} \cdot \mathbf{H}) = 0$. Here, too, the mapped divergence-free conditions allow the identification of $\epsilon \mathcal{T}$ and $\mu \mathcal{T}$ as uniaxial electric permittivity and magnetic permeability tensors, respectively, thus facilitating the derivation in section 3 of the hyperbolic equations to implement the layer in the time domain.

The transition from Ω_c to Ω_ρ at $\rho' = \rho_0$ will be reflectionless since (2.37) enforces continuity of the unprimed tangential and primed normal electromagnetic field components (recall $\lim_{u \rightarrow u_0^+} \gamma_u = 1$ and $u = \rho$ here). This assertion is rigorously proven in the same way as in section 2.1, Case a. The solution of (2.38) is obtained by applying (2.37) to the solution of (2.36), which is expressed as a linear combination of electric and magnetic multipole fields in spherical coordinates [8]. Using (2.39) and the corresponding relation for the magnetic field, it is easy to derive scalar Helmholtz equations satisfied by the mapped $(\rho \hat{a}_{\rho'}) \cdot \mathbf{E}'$ and $(\rho \hat{a}_{\rho'}) \cdot \mathbf{H}'$ in Ω_ρ . For example, we find that the electric multipole potential $U^e(\rho, \theta, \phi) = \rho \gamma_\rho \zeta_\rho E_\rho$ (i.e., the mapped $(\rho \hat{a}_{\rho'}) \cdot \mathbf{E}'$) satisfies

$$(2.40) \quad \begin{aligned} \frac{1}{\rho \gamma_\rho} \zeta_\rho \frac{\partial}{\partial \rho} \left[\zeta_\rho \frac{\partial}{\partial \rho} (\rho \gamma_\rho U^e) \right] &+ \frac{1}{\rho^2 \gamma_\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U^e}{\partial \theta} \right) \\ &+ \frac{1}{\rho^2 \gamma_\rho^2 \sin^2 \theta} \frac{\partial^2 U^e}{\partial \phi^2} + \omega^2 \epsilon \mu U^e = 0. \end{aligned}$$

Therefore, U^e is given by an infinite series in a linear combination of products of incoming/outgoing spherical Hankel functions ($h_l^{(1,2)}(k\rho)$) and the spherical harmonics ($Y_{lm}(\theta, \phi)$). This is similar for the magnetic multipole potential U^m in Ω_ρ . An arbitrary propagating field in the sponge region is then expressed as a linear combination of the fields generated by the multipoles, and standard analysis easily leads to the proof that the interface $\partial\Omega_c$ is reflectionless. Also, waves in Ω_ρ decay as required, i.e., exponentially in the $\hat{\rho}$ -direction and independently of ω . Each outgoing component of the multipole expansion of the solution in Ω_ρ is proportional to spherical Hankel

functions of the first kind and of real order l , so it will suffice to determine how (2.37) alters their behavior. It is a simple matter to determine that the waves decay as required in Ω_ρ since

$$(2.41) \quad |h_l^{(1)}(\eta\rho')| \approx \frac{|C_l|}{\eta|\rho'|} e^{-\eta\rho'_i}, \quad |\rho'| \rightarrow \infty,$$

where now $\eta = \omega\sqrt{\epsilon\mu}\hat{\rho} \cdot \mathbf{k}$, and ρ'_i is of the form given below (2.20). Waves moving in the layer toward $\rho = \rho_0$ involve $h_l^{(2)}(\eta\rho')$ and will be similarly damped. The layer is terminated with a Dirichlet condition applied to the tangential fields E_ϕ and E_θ at $\rho = \rho_1$, and the resultant reflection coefficient is again exponentially small.

3. Hyperbolicity, well-posedness, and causality in the time domain.

The identification of $\epsilon\mathcal{T}$ and $\mu\mathcal{T}$ as uniaxial permittivity and permeability tensors allows the introduction of the constitutive electric and magnetic relations $\mathbf{D}(\mathbf{x}, \omega) = \epsilon\mathcal{T}(\mathbf{x}, \omega) \cdot \mathbf{E}(\mathbf{x}, \omega)$ and $\mathbf{B}(\mathbf{x}, \omega) = \mu\mathcal{T}(\mathbf{x}, \omega) \cdot \mathbf{H}(\mathbf{x}, \omega)$, where $\mathcal{T}$ is the diagonal “material” matrix appearing in the transformed (2.1) in each coordinate system and subregion. With simple algebra each $\mathcal{T}$ can be decomposed as $\mathcal{T} = \mathcal{T}_\xi + \mathcal{T}_\omega$, where $\mathcal{T}_\xi$ is a diagonal real matrix whose elements depend only on the ξ_u (independent of ω) and satisfy $\lim_{\xi_u \rightarrow 1} \mathcal{T}_\xi = \mathbf{I}$, and $\mathcal{T}_\omega$ is a diagonal complex matrix whose elements are proper rational functions of ω that vanish as $O(\frac{1}{\omega})$ for $\omega \rightarrow \infty$. This results in the constitutive laws being of the form $\mathbf{D}(\mathbf{x}, \omega) = \epsilon\mathcal{T}_\xi \cdot \mathbf{E}(\mathbf{x}, \omega) + \mathbf{P}$ and $\mathbf{B}(\mathbf{x}, \omega) = \mu\mathcal{T}_\xi \cdot \mathbf{H}(\mathbf{x}, \omega) + \mathbf{M}$. Consequently, the inverse Fourier transforms of the electromagnetic variables $\mathbf{P} = \epsilon\mathcal{T}_\omega(\mathbf{x}, \omega) \cdot \mathbf{E}(\mathbf{x}, \omega)$ and $\mathbf{M} = \mu\mathcal{T}_\omega(\mathbf{x}, \omega) \cdot \mathbf{H}(\mathbf{x}, \omega)$, i.e., the induced polarization functions with a parametric dependence on the spatial variables, satisfy ordinary differential equations in time forced by the $\mathbf{E}$ and $\mathbf{H}$ fields (that is, convolution is not necessary). Further, being lower-order terms their contributions to the systems of equations in each subregion can be dropped for analysis [7]; i.e., hyperbolicity, causality, and well-posedness of the Cauchy problem depend only on the relevant $\mathcal{T}_\xi$ (which influences the principal part only) in each coordinate system and subregion. The complete 6×6 Maxwell system that holds in each case can be cast in the coordinate-independent form

$$(3.1) \quad \begin{pmatrix} \frac{\partial \mathbf{E}}{\partial t} \\ \frac{\partial \mathbf{H}}{\partial t} \end{pmatrix} + \begin{pmatrix} -\frac{\mathcal{T}_\xi^{-1} \cdot \nabla \times \mathbf{H}}{\epsilon} \\ \frac{\mathcal{T}_\xi^{-1} \cdot \nabla \times \mathbf{E}}{\mu} \end{pmatrix} + \begin{pmatrix} \mathcal{T}_\xi^{-1} \cdot \mathcal{F}^{-1}\{\mathcal{T}_\omega\} * \mathbf{E} \\ \mathcal{T}_\xi^{-1} \cdot \mathcal{F}^{-1}\{\mathcal{T}_\omega\} * \mathbf{H} \end{pmatrix} = 0,$$

where the symbol $*$ denotes convolution, and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The reader is reminded that due to the special form of $\mathcal{T}_\omega$ convolution is not necessary; rather, one defines a set of 6 auxiliary functions (the $\mathbf{P}, \mathbf{M}$) to represent the convolutions and appends to the Maxwell system 6 ordinary differential equations to compute their time evolution.

We now provide the relevant theoretical details using, for the sake of brevity, the two-dimensional equations in the Ω_z subregion in rectangular coordinates (see Figure A.1 and the appendix).

In the TE-polarization, the two-dimensional electromagnetic field consists of the vector $\mathbf{U}(x, z, \omega) = (E_x, E_z, H_y)^T$. Applying the inverse Fourier transform in the region $z < z_0$ (in Ω_c) we obtain the system satisfied by the vector function $\mathbf{U}(x, z, t) =$

$\mathcal{F}^{-1}\{\mathbf{U}(x, z, \omega)\}$:

$$(3.2) \quad \frac{\partial \mathbf{U}}{\partial t} + \begin{pmatrix} 0 & 0 & \frac{1}{\epsilon} \\ 0 & 0 & 0 \\ \frac{1}{\mu} & 0 & 0 \end{pmatrix} \frac{\partial \mathbf{U}}{\partial z} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu} \\ 0 & -\frac{1}{\epsilon} & 0 \end{pmatrix} \frac{\partial \mathbf{U}}{\partial x} = 0.$$

The eigenvalues of the matrices in (3.2) are the wave speeds $\{-c, 0, c\}$, where $c = \frac{1}{\sqrt{\epsilon\mu}}$ is the wavefront speed in the dielectric. There is one distinct eigenvector for each eigenvalue; the system is strongly hyperbolic and causal. The change of variables $\mathbf{V} = \mathcal{C}^{-1} \cdot \mathbf{U}$, where $\mathcal{C} = \text{diag}\{1, 1, \sqrt{\frac{\epsilon}{\mu}}\}$, shows that (3.2) is symmetric and therefore strongly well-posed [24].

In the region $z \geq z_0$ (in Ω_z), the relevant $\mathcal{T}_\xi$ is $\mathcal{T}_\xi = \text{diag}\{\xi_z, \xi_z, \frac{1}{\xi_z}\}$, and the principal part of the time-domain sponge layer equations is

$$(3.3) \quad \frac{\partial \mathbf{U}}{\partial t} + \begin{pmatrix} 0 & 0 & \frac{1}{\xi_z \epsilon} \\ 0 & 0 & 0 \\ \frac{1}{\xi_z \mu} & 0 & 0 \end{pmatrix} \frac{\partial \mathbf{U}}{\partial z} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\xi_z}{\epsilon} \\ 0 & -\frac{1}{\xi_z \mu} & 0 \end{pmatrix} \frac{\partial \mathbf{U}}{\partial x}.$$

When $\xi_z = 1$, (3.3) is identical to (3.2); thus it is the principal part of a causal strongly well-posed hyperbolic system and the full set of equations is a causal strongly well-posed hyperbolic system. Choosing $\xi_z > 1$ results in anisotropy; the eigenvalues of (3.3) in the $\hat{z}$ -direction are then given by the triplet $\{-\frac{c}{\xi_z}, 0, \frac{c}{\xi_z}\}$ (slow-down), while in the $\hat{x}$ -direction they are given by $\{-c, 0, c\}$. Each coefficient matrix in (3.3) has a full set of eigenvectors; therefore it is the principal part of a strongly hyperbolic causal system. Causality is preserved because the maximum wave speed in Ω_z , which occurs in the $\hat{x}$ -direction, is c ($> \frac{c}{\xi_z}$).

We now establish that (3.3), with $\xi_z > 1$, is the principal part of a strongly well-posed problem by showing that it is symmetric; i.e., its coefficient matrices can be simultaneously symmetrized by some nonsingular similarity transformation. Applying R_z , the diagonalizer of the first matrix A in (3.3), and noting it is *identical* to the diagonalizer of the first matrix in (3.2) and independent of ξ_z , i.e., $\Lambda_z = \text{diag}\{-\frac{c}{\xi_z}, 0, \frac{c}{\xi_z}\} = R_z^{-1} A R_z$, to the second matrix B in (3.3) we obtain

$$(3.4) \quad R_z^{-1} B R_z = \begin{pmatrix} 0 & -\frac{1}{2\mu\xi_z} & 0 \\ -\frac{\xi_z}{\epsilon} & 0 & -\frac{\xi_z}{\epsilon} \\ 0 & -\frac{1}{2\mu\xi_z} & 0 \end{pmatrix}.$$

To show that A and B can be simultaneously symmetrized, we seek a diagonal matrix D , such that $D^{-1} R_z^{-1} B R_z D$ is symmetric, and note $D^{-1} \Lambda_z D$ remains diagonal (hence symmetric). Simple algebra determines that $D = \text{diag}\{1, \sqrt{2}\xi_z \sqrt{\frac{\mu}{\epsilon}}, 1\}$ is an appropriate choice. We have just shown that (3.3) is symmetric; hence it is the principal part of a strongly well-posed causal hyperbolic system. Thus, (3.1) in Ω_z , including the lower-order term, is a causal strongly well-posed hyperbolic system and does not possess the undesirable properties of the Berenger system.

The time-domain form of the equations in all other subregions and coordinate systems is obtained through the relevant constitutive law for the particular layer and the inverse Fourier transform. We indicate how our sponge layer would be implemented in three-dimensional cylindrical coordinates for transient waves by giving the hyperbolic Ampere's law that results from (2.7). For simplicity we set $\xi_\rho = 1$ and

define $\bar{\sigma}_\rho(\rho) = \frac{1}{\rho} \int_{\rho_0}^{\rho} \sigma_\rho(s) ds$. In $\rho_0 \leq \rho < \rho_1$ we obtain

$$(3.5) \quad \begin{aligned} \frac{\partial D_\rho}{\partial t} &= (\nabla \times \mathbf{H})_\rho, & \frac{dD_\rho}{dt} + \sigma_\rho(\rho) D_\rho &= \epsilon \frac{dE_\rho}{dt} + \epsilon \bar{\sigma}_\rho(\rho) E_\rho, \\ \frac{\partial D_\phi}{\partial t} &= (\nabla \times \mathbf{H})_\phi, & \frac{dD_\phi}{dt} + \bar{\sigma}_\rho(\rho) D_\phi &= \epsilon \frac{dE_\phi}{dt} + \epsilon \sigma_\rho(\rho) E_\phi, \\ \frac{\partial D_z}{\partial t} + \sigma_\rho(\rho) D_z &= (\nabla \times \mathbf{H})_z, & \frac{dD_z}{dt} &= \epsilon \frac{dE_z}{dt} + \epsilon \bar{\sigma}_\rho(\rho) E_z, \end{aligned}$$

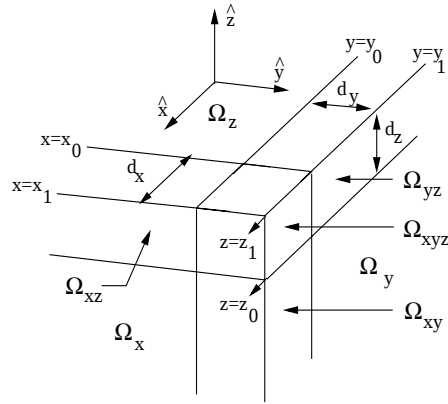
and similarly for Faraday's law. In a computational setting, 6 ordinary differential equations are solved at each spatial point in the layer, and the Dirichlet condition (or some other ABC) is imposed on E_ϕ and E_z at $\rho = \rho_1$ to truncate the layer. That (3.5), along with Faraday's law, is a causal strongly well-posed hyperbolic system is immediately obvious, once the lower-order terms are dropped through setting $\sigma_\rho = 0$. Similar considerations apply to the hyperbolic systems obtained when the inverse Fourier transform is applied to the $\Omega_{z\rho}$ and Ω_z equations (i.e., Maxwell in a "material" described by (2.32) and (A.4), respectively), and to (2.38) in spherical coordinates.

4. Summary and discussion. In this paper we have presented a very effective alternative to radiation boundary conditions for the numerical solution of the three-dimensional Maxwell equations in rectangular, cylindrical, and spherical coordinates. The numerical result in cylindrical coordinates presented in section 2 should be viewed as a lower bound on the error that will be obtained if the problem was solved with a numerical scheme since any discretization error would be additional to the error shown there. Thus, in a discrete setting the size of the damping D controls the spatial mesh size. The boundary truncation presented herein is independent of the approach used to effect the spatial discretization and can be applied with equal ease to frequency-domain (elliptic) and time-domain (hyperbolic) problems solved with finite element, finite difference, and spectral methods.

Although we developed the approach to the derivation of reflectionless sponge layers for use in the numerical solution of scattering problems embedded in a homogeneous dielectric space, the general spirit of our approach is not confined to such problems. It would be trivial to derive layers for use in simulations where the background medium is a layered medium. For example, let the Ω_z region cut perpendicularly through the layering (i.e., the $\hat{z}$ -axis is orthogonal to the direction of the layering); in this case one would derive layer equations for each dielectric layer as though it filled all of the space. Consequently, the electric and magnetic "material" matrices in each layer would be $\epsilon_j \mathcal{T}$ and $\mu_j \mathcal{T}$, respectively, where j indexes the layers. This is similar for other waveguiding environments in cylindrical coordinates.

We close by mentioning recent numerical results [22] which indicate our reflectionless sponge layer provides levels of numerical reflection from $\partial\Omega_c$ that are comparable to those obtained with the *exact* ABC [14] for vector spherical waves scattering from a dielectric sphere but at a substantial savings in computational cost. We are presently investigating this issue by concentrating on analysis of the *discrete* reflectionless sponge layers presented herein and on accuracy and computational cost comparisons with available exact and (high-order) local ABCs for each coordinate system.

Appendix A. For completeness, we now give our approach in the three-dimensional rectangular coordinate system.

FIG. A.1. Rectangular coordinates; Ω_c is occluded by the vertex geometry.

A.1. Ω_z . Consider the region $z' \geq z_0$, depicted as Ω_z in Figure A.1. The appropriate scaling is

$$(A.1) \quad \begin{aligned} \mathbf{x}' &= \begin{cases} \mathbf{x}; & z' < z_0, \\ \text{diag}\{1, 1, \gamma_z\} \cdot \mathbf{x}; & z' \geq z_0, \end{cases} \\ \mathbf{E}' &= \begin{cases} \mathbf{E}; & z' < z_0, \\ \text{diag}\{1, 1, \zeta_z\} \cdot \mathbf{E}; & z' \geq z_0, \end{cases} \\ \mathbf{H}' &= \begin{cases} \mathbf{H}; & z' < z_0, \\ \text{diag}\{1, 1, \zeta_z\} \cdot \mathbf{H}; & z' \geq z_0, \end{cases} \end{aligned}$$

and the curl of the electric field in (2.1), for example, transforms as

$$(A.2) \quad \nabla' \times \mathbf{E}' = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'} \\ E_{x'} & E_{y'} & E_{z'} \end{vmatrix} \rightarrow \begin{pmatrix} \zeta_z & 0 & 0 \\ 0 & \zeta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \nabla \times \mathbf{E}.$$

This is similar for $\nabla' \times \mathbf{H}'$. In the context of the frequency-domain Maxwell equations, the scale factor, appearing in (A.2) as a coefficient of the curl in the unprimed variables, can be identified with a dielectric permeability since it is recognizable as the negative of the r.h.s. in Faraday's law in the unprimed variables. Applying (A.1) to (2.1) in Ω_z we obtain

$$(A.3) \quad \begin{aligned} -i\omega\epsilon \begin{pmatrix} \frac{1}{\zeta_z} & 0 & 0 \\ 0 & \frac{1}{\zeta_z} & 0 \\ 0 & 0 & \zeta_z \end{pmatrix} \cdot \mathbf{E} &= \nabla \times \mathbf{H}, \\ -i\omega\mu \begin{pmatrix} \frac{1}{\zeta_z} & 0 & 0 \\ 0 & \frac{1}{\zeta_z} & 0 \\ 0 & 0 & \zeta_z \end{pmatrix} \cdot \mathbf{H} &= -\nabla \times \mathbf{E}. \end{aligned}$$

Thus, scaling in the frequency domain is seen to be equivalent to mapping an isotropic dielectric with constitutive relations $\mathbf{D}'(\mathbf{B}') = \epsilon(\mu)\mathbf{E}'(\mathbf{H}')$ to a dielectric that is in-

homogeneous and uniaxial anisotropic with constitutive relations

$$(A.4) \quad \mathbf{D}(\mathbf{B}) = \epsilon(\mu)\mathcal{T} \cdot \mathbf{E}(\mathbf{H}) = \epsilon(\mu) \begin{pmatrix} \frac{1}{\zeta_z} & 0 & 0 \\ 0 & \frac{1}{\zeta_z} & 0 \\ 0 & 0 & \zeta_z \end{pmatrix} \cdot \mathbf{E}(\mathbf{H}).$$

Following the application of the particular scaling, the diagonal tensor $\mathcal{T}$ in (A.4) can then be defined a posteriori for all cases in this appendix.

The plane $z' = z_0$ will be reflectionless for waves impinging upon it from the region $z' < z_0$ because (A.1) enforces continuity of *all* the electromagnetic field components (tangential and normal) in the primed variables; primed and unprimed tangential-to- $\hat{z}$ electromagnetic components are continuous, and so are the primed normal components of displacement $[D'_{z'}]_{z'=z_0} = 0$. The plane-wave scattering calculations in [2], using (A.4), prove the reflectionless property.

We now determine the nature of the waves in Ω_z in order to show that the “dielectric” defined by (A.4) is lossy. The zero-divergence condition for the electric field in the primed coordinate system transforms under (A.1) in Ω_z as

$$(A.5) \quad \nabla' \cdot (\epsilon \mathbf{E}') = 0 \rightarrow \epsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \zeta_z \frac{\partial (\zeta_z E_z)}{\partial z} \right) = \nabla \cdot (\epsilon \mathcal{T} \cdot \mathbf{E}) = 0.$$

With (A.5) and (A.3), the following Helmholtz equation for the components of $\mathbf{E}$ are obtained:

$$(A.6) \quad \begin{aligned} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial E_x}{\partial z} \right) + \omega^2 \epsilon \mu E_x &= 0, \\ \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial E_y}{\partial z} \right) + \omega^2 \epsilon \mu E_y &= 0, \\ \frac{\partial^2}{\partial x^2} (\zeta_z E_z) + \frac{\partial^2}{\partial y^2} (\zeta_z E_z) + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial}{\partial z} (\zeta_z E_z) \right) + \omega^2 \epsilon \mu (\zeta_z E_z) &= 0. \end{aligned}$$

Using the zero-divergence condition for the magnetic field in the primed coordinate system and (A.1), we can obtain the corresponding Helmholtz equations for the components of $\mathbf{H}$ in Ω_z . These will be similar to (A.6) with the appropriate dependent variable changes.

We briefly consider now the effect of the choice for $\alpha_u(s, \omega)$ in the case of transient fields with zero-frequency content. Substituting (2.4) in the third equation of (A.6) we obtain

$$(A.7) \quad \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\omega^2}{\xi_z^2} \frac{\partial}{\partial z} \left[\frac{1}{\omega + i\sigma_z(z)} \frac{\partial}{\partial z} \left(\frac{E_z}{\omega + i\sigma_z(z)} \right) \right] + \omega^2 \epsilon \mu E_z = 0.$$

If, in a transient simulation, E_z has zero-frequency content (A.7) shows these components (at $\omega = 0$) do not satisfy the standard three-dimensional Laplace equation and the simulation will be erroneous. However, this is not an issue for scattering problems as freely propagating fields generated by real sources do not have zero-frequency content. In any case, one may be interested in solving contrived problems where the fields have zero-frequency content, e.g., due to a source whose amplitude rises smoothly from zero to some constant value that it maintains thereafter. Clearly, the late-time fields in the layer will be inaccurate for such an illumination. The alternative

choice for $\alpha_u(s, \omega)$ given in section 2 results in

$$(A.8) \quad \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{(1-i\omega)^2}{\xi_z^2} \frac{\partial}{\partial z} \left[\frac{1}{1-i\omega+\sigma_z(z)} \frac{\partial}{\partial z} \left(\frac{E_z}{1-i\omega+\sigma_z(z)} \right) \right] + \omega^2 \epsilon \mu E_z = 0,$$

which, at $\omega = 0$, reduces to the standard Laplace equation with a stretched $\hat{z}$ -coordinate according to $z' = z_0 + \int_{z_0}^z \xi_z(1 + \sigma_z(s)) ds$. The layer, in this case, correctly accounts for the zero-frequency content.

The observation that (A.6), in light of (A.1) with (2.4), are equivalent to the standard Helmholtz equation satisfied by each component of $\mathbf{E}'$ leads to each equation in (A.6) being equivalent to the corresponding component of

$$(A.9) \quad \frac{\partial^2 \mathbf{E}'}{\partial x'^2} + \frac{\partial^2 \mathbf{E}'}{\partial y'^2} + \frac{\partial^2 \mathbf{E}'}{\partial z'^2} + \omega^2 \epsilon \mu \mathbf{E}' = 0.$$

This is similar for the magnetic fields. The plane wave solutions of (A.9) are

$$(A.10) \quad \mathbf{E}' = \tilde{\mathbf{E}}'_0 e^{\pm i\omega \sqrt{\epsilon \mu} \mathbf{k} \cdot \mathbf{x}'},$$

where $\tilde{\mathbf{E}}'_0$ is a complex constant vector, $\mathbf{k}$ is the unit vector in the direction of propagation of the plane wave, and the $\pm$ corresponds to right/left moving waves with respect to the direction of $\mathbf{k}$. The corresponding magnetic field is obtained from Faraday's law in (2.1). Therefore, the solutions of (A.6) can be obtained by applying (A.1) to (A.10), and then, in Ω_z , the transverse-to- $\hat{z}$ wavenumbers are preserved across $z' = z_0$ while in the $\hat{z}$ -direction we obtain

$$(A.11) \quad e^{\pm i\omega \sqrt{\epsilon \mu} \kappa_z z'} \rightarrow e^{\pm i\omega \sqrt{\epsilon \mu} \kappa_z (z_0 + \xi_z(z - z_0)) \mp \sqrt{\epsilon \mu} \kappa_z \xi_z \int_{z_0}^z \sigma_z(s) ds},$$

with $\kappa_z = \hat{z} \cdot \mathbf{k}$, i.e., frequency-independent exponential decay in the $\hat{z}$ -direction for all fields in Ω_z . We must point out that (A.11) indicates that both right/left moving waves are exponentially damped. Assuming that Ω_z is terminated at $z = z_1$ (layer of width $d_z = z_1 - z_0$) with a Dirichlet condition on the tangential fields E_x and E_y , the magnitude of the reflection from the layer, due to a plane wave of amplitude $\tilde{\mathbf{E}}'_0$ incident on $\partial\Omega_c$ from Ω_c , is

$$(A.12) \quad |R_{\Omega_z}| = e^{-2\sqrt{\epsilon \mu} \kappa_z \xi_z \int_{z_0}^{z_1} \sigma_z(s) ds}.$$

This is in accordance with Berenger's result [1] and is similar to (2.22).

In the Ω_x and Ω_y regions, depicted in Figure A.1, scalings similar to (A.1) would be used to determine the appropriate sponge layer equations. The resultant systems, along with those in Ω_z and a Dirichlet boundary condition on the tangential fields at the outer surfaces, are used to close the six faces of the computational box Ω_c .

A.2. Ω_{yz} . Consider next the region $y' \geq y_0$, $z' \geq z_0$, depicted as Ω_{yz} in Figure A.1, where, with the appropriate scaling,

$$(A.13) \quad \begin{aligned} \mathbf{x}' &= \text{diag}\{1, \gamma_y, \gamma_z\} \cdot \mathbf{x}, \\ \mathbf{E}' &= \text{diag}\{1, \zeta_y, \zeta_z\} \cdot \mathbf{E}, \\ \mathbf{H}' &= \text{diag}\{1, \zeta_y, \zeta_z\} \cdot \mathbf{H}. \end{aligned}$$

Equation (2.1) transforms to a system like (A.3) but with the material tensor $\mathcal{T}$ now being

$$(A.14) \quad \mathcal{T} = \begin{pmatrix} \frac{1}{\zeta_y \zeta_z} & 0 & 0 \\ 0 & \frac{\zeta_y}{\zeta_z} & 0 \\ 0 & 0 & \frac{\zeta_z}{\zeta_y} \end{pmatrix}.$$

Constitutive relations in the form of (A.4) can again be defined. Since (A.13) enforces continuity of the unprimed tangential and primed normal field components across the transition from Ω_z (see (A.1)) to Ω_{yz} at $y' = y_0$, and across the transition from Ω_y to Ω_{yz} (using the appropriate scaling for this region) at $z' = z_0$, these transitions are reflectionless.

We must now show that the waves in Ω_{yz} are damped, in both $\hat{y}$ - and $\hat{z}$ -directions, independently of the frequency. We do this by again appealing to the zero-divergence conditions. For the electric field in the primed coordinate system we have, under (A.13),

$$(A.15) \quad \nabla' \cdot (\epsilon \mathbf{E}') = 0 \rightarrow \epsilon \left(\frac{\partial E_x}{\partial x} + \zeta_y \frac{\partial(\zeta_y E_y)}{\partial y} + \zeta_z \frac{\partial(\zeta_z E_z)}{\partial z} \right) = \nabla \cdot (\epsilon \mathcal{T} \cdot \mathbf{E}) = 0.$$

With (A.15), the following Helmholtz equations are obtained for the electric fields in Ω_{yz} :

$$(A.16) \quad \begin{aligned} \frac{\partial^2 E_x}{\partial x^2} + \zeta_y \frac{\partial}{\partial y} \left(\zeta_y \frac{\partial E_x}{\partial y} \right) + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial E_x}{\partial z} \right) + \omega^2 \epsilon \mu E_x &= 0, \\ \frac{\partial^2(\zeta_y E_y)}{\partial x^2} + \zeta_y \frac{\partial}{\partial y} \left(\zeta_y \frac{\partial(\zeta_y E_y)}{\partial y} \right) + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial(\zeta_y E_y)}{\partial z} \right) + \omega^2 \epsilon \mu(\zeta_y E_y) &= 0, \\ \frac{\partial^2(\zeta_z E_z)}{\partial x^2} + \zeta_y \frac{\partial}{\partial y} \left(\zeta_y \frac{\partial(\zeta_z E_z)}{\partial y} \right) + \zeta_z \frac{\partial}{\partial z} \left(\zeta_z \frac{\partial(\zeta_z E_z)}{\partial z} \right) + \omega^2 \epsilon \mu(\zeta_z E_z) &= 0. \end{aligned}$$

Equations (A.16) are equivalent to (A.9) under (A.13) in Ω_{yz} . Again, their solutions can be obtained by applying (A.13) (with (2.4)) to (A.10), and the waves will now decay in space exponentially

$$(A.17) \quad e^{-\sqrt{\epsilon \mu} \kappa_y \xi_y \int_{y_0}^y \sigma_y(s) ds - \sqrt{\epsilon \mu} \kappa_z \xi_z \int_{z_0}^z \sigma_z(s) ds},$$

where $\kappa_y = \hat{y} \cdot \mathbf{k}$, as they move outward in Ω_{yz} . This is similar for the magnetic field.

In the regions designated on Figure A.1 as Ω_{xy} and Ω_{xz} , one would use the scalings similar to (A.13) in order to derive the appropriate equations. The resulting systems, along with those in Ω_{yz} and a Dirichlet condition on the tangential fields at the outer surfaces, are used to close the 12 edges of the computational box Ω_c .

A.3. Ω_{xyz} . Finally, consider the region $x' \geq x_0$, $y' \geq y_0$, $z' \geq z_0$, depicted as Ω_{xyz} in Figure A.1, for which the appropriate scaling is

$$(A.18) \quad \begin{aligned} \mathbf{x}' &= \text{diag}\{\gamma_x, \gamma_y, \gamma_z\} \cdot \mathbf{x}, \\ \mathbf{E}' &= \text{diag}\{\zeta_x, \zeta_y, \zeta_z\} \cdot \mathbf{E}, \\ \mathbf{H}' &= \text{diag}\{\zeta_x, \zeta_y, \zeta_z\} \cdot \mathbf{H}. \end{aligned}$$

Under (A.18), (2.1) is transformed to a system like (A.3) but with the material tensor $\mathcal{T}$ now being

$$(A.19) \quad \mathcal{T} = \begin{pmatrix} \frac{\zeta_x}{\zeta_y \zeta_z} & 0 & 0 \\ 0 & \frac{\zeta_y}{\zeta_x \zeta_z} & 0 \\ 0 & 0 & \frac{\zeta_z}{\zeta_x \zeta_y} \end{pmatrix}.$$

Thus, constitutive relations like (A.4) can also be defined in this region. Some thought leads to the realization that (A.18) enforces continuity of the unprimed tangential and primed normal field components across the transition from Ω_{xy} (with the appropriate scaling for this region) to Ω_{xyz} at $z' = z_0$, across the transition from Ω_{xz} (with the appropriate scaling for this region) to Ω_{xyz} at $y' = y_0$, and across the transition from Ω_{yz} (see (A.13)) to Ω_{xyz} at $x' = x_0$; these transitions are reflectionless.

The zero-divergence condition for $\mathbf{E}'$ in Ω_{xyz} now transforms as

$$(A.20) \quad \begin{aligned} \nabla' \cdot (\epsilon \mathbf{E}') &= 0 \\ \rightarrow \epsilon \left(\zeta_x \frac{\partial(\zeta_x E_x)}{\partial x} + \zeta_y \frac{\partial(\zeta_y E_y)}{\partial y} + \zeta_z \frac{\partial(\zeta_z E_z)}{\partial z} \right) &= \nabla \cdot (\epsilon \mathcal{T} \cdot \mathbf{E}) = 0 \end{aligned}$$

and allows us to derive the following Helmholtz equations for the electric fields:

$$(A.21) \quad \sum_{l=x,y,z} \zeta_l \frac{\partial}{\partial l} \left(\zeta_l \frac{\partial(\zeta_n E_n)}{\partial l} \right) + \omega^2 \epsilon \mu (\zeta_n E_n) = 0, \quad n = x, y, z.$$

Equations (A.21), and the corresponding ones for $\mathbf{H}$, are equivalent to (A.9) (and to the corresponding one for $\mathbf{H}'$) under (A.18). Thus, their solutions can again be obtained by applying (A.18) to (A.10), and the outward-moving waves will now decay in space exponentially as

$$(A.22) \quad e^{-\sqrt{\epsilon \mu} \kappa_x \xi_x \int_{x_0}^x \sigma_x(s) ds - \sqrt{\epsilon \mu} \kappa_y \xi_y \int_{y_0}^y \sigma_y(s) ds - \sqrt{\epsilon \mu} \kappa_z \xi_z \int_{z_0}^z \sigma_z(s) ds},$$

where $\kappa_x = \hat{x} \cdot \mathbf{k}$, independently of the frequency.

A Dirichlet boundary condition on the tangential fields at the outer surfaces is used to close all 8 vertices of the computational box, Ω_c .

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